

Dixmier's Problem 5 for the Weyl Algebra

V. V. Bavula

Abstract

In the paper [16], J. Dixmier posed six problems for the first Weyl algebra. In this paper we give a solution to the Dixmier's Problem 5 from this paper. Problem 3 was solved by Joseph and Stein [21] (using results of McConnell and Robson [25]). Using a (difficult) polarization theorem for the first Weyl algebra Joseph [21] solved problem 6 (a short proof to this problem is given in [7], note that the same result and the proof are true for the ring of differential operators on an arbitrary smooth irreducible algebraic curve [7]). Problems 1, 2, and 4 are still open.

1 Introduction

Let K be a field of characteristic zero. The *first Weyl algebra* A_1 is an associative K -algebra generated over K by elements X and Y subject to the defining relation $YX - XY = 1$. The n 'th *Weyl algebra* A_n is the tensor product $A_1 \otimes \cdots \otimes A_1$ of n copies of the first Weyl algebra. The Weyl algebra A_n is a simple Noetherian domain of Gelfand-Kirillov dimension $2n$ which is canonically isomorphic to the ring of differential operators $K[X_1, \dots, X_n, \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}]$ with polynomial coefficients. The Weyl algebras have been intensively studied during the last fifty years. The Gelfand-Kirillov dimension and the transcendence dimension of the Weyl algebra A_n were computed by Gelfand and Kirillov, [19]. The fact that each derivation of the Weyl algebra A_n is an inner derivation was proved by Dixmier, [18]. The commutativity of the centralizer of an arbitrary nonzero element of the first Weyl algebra was proved by Amitsur, [1]. The structure of maximal commutative subalgebras of the first Weyl algebra was studied and the generators of the group of all algebra automorphisms of A_1 were found by Dixmier, [16] (see [8] for a generalization of these results to noncommutative deformations of type- A Kleinian singularities). Simple A_1 -modules were classified by Block, [13] (see also [4] for an alternative approach and some generalizations). The global dimension of A_n is n , this was proved by Rinehart, [28] in the case $n = 1$, and by Roos in the general case. Rentschler and Gabriel proved that the Krull dimension of A_n is n , [27]. The finite dimensionality of the vector spaces $\text{Ext}_{A_1}^i$ and $\text{Tor}_i^{A_1}$ for simple A_1 -modules was established by McConnell and Robson, [25]. The fact that the Gelfand-Kirillov dimension of a nonzero finitely generated A_n -module is not less than n (*the Bernstein Inequality*) was proved by Bernstein, [10]. A finitely generated A_n -module of Gelfand-Kirillov dimension n is called a *holonomic* module. Each simple module over

the first Weyl algebra is holonomic. The situation is completely different for the Weyl algebras A_n , $n \geq 2$. The first examples of *non-holonomic* A_n -modules were constructed by Stafford, [30], further progress in this direction was made by Coutinho, [14]. Bernstein and Lunts, [11] in the case of the second Weyl algebra, and Lunts in the general case, [23], showed that “generically” a simple A_n -module is non-holonomic and has Gelfand-Kirillov dimension $2n - 1$. Simple holonomic A_2 -modules were classified by the author and van Oystaeyen, [9]. Skew subfields of the n 'th Weyl skew field which are invariant under the action of a finite group were studied by Alev and Dumas, [2]. Makar-Limanov, [24], proved that the first Weyl skew field contains a free subalgebra.

In his fundamental paper [16] Dixmier initiated a systematic study of the structure of the first Weyl algebra A_1 . At the end of his paper he posed 6 problems. In this paper a *negative* answer is given to the 5'th problem. *Problem 1* concerns the question *whether an algebra endomorphism of A_1 is an algebra automorphism?* A positive answer to a similar problem but for the n 'th Weyl algebra implies the *Jacobian Conjecture* as was shown by Bass, Connell and Wright, [3]. For an arbitrary non-scalar element u of A_1 one can associate the *inner derivation* $\text{ad } u$ of the Weyl algebra A_1 , $\text{ad } u(a) = ua - au$, $a \in A_1$, and then the \mathbb{N} -filtered algebra $N(u) = \cup_{i \geq 0} N(u, i)$ where $N(u, i) := \ker(\text{ad } u)^{i+1}$. The zero component of this filtration, $\ker \text{ad } u$, is the *centralizer* $C(u)$ of the element u in A_1 . The algebra $C(u)$ is a commutative algebra which is a free finitely generated module over its polynomial subalgebra $K[u]$, [1] and [16]. Dixmier partitioned all non-scalar elements of the Weyl algebra A_1 into 5 classes $\Delta_1, \dots, \Delta_5$, and classified up to the action of the group $\text{Aut}_K(A_1)$ elements from the class Δ_1 , so-called *elements of strongly nilpotent type*, and elements from the class Δ_3 , so-called *elements of strongly semi-simple type*. Problems 2-6 are concerned with properties and classification of elements from the remaining classes Δ_2 , Δ_4 and Δ_5 . A non-scalar element $u \in A_1$ with $C(u) \neq N(u)$ (resp., with $N(u) = A_1$) is called an element of *nilpotent type* (resp., *of strongly nilpotent type*). A non-scalar element $u \in A_1$ of nilpotent type belongs to Δ_2 iff $C(u) \neq N(u) \neq A_1$, and we say that u is of *weakly nilpotent type*.

Dixmier's Problem 5, [16]: *Let $u \in A_1$ be an element of nilpotent type. Set $I_n = (\text{ad } u)^n N(u, n)$; this an ideal of $C(u)$. Is $I_{n+1} = I_1 I_n$ for n sufficiently large?*

The next Theorem shows that the answer in general is negative. This result is proved in Section 4, Corollary 4.2

For given natural numbers n and $m \neq 0$, there exist and unique natural numbers l and r such that $n = lm + r$ and $0 \leq r < m$. The number l is denoted by $[\frac{n}{m}]$.

Theorem 1.1 *Let $\alpha(H) \in K[H]$ be a polynomial of degree $d \geq 1$ in the variable $H = YX$. The centralizer of the element $u = \alpha(YX)X \in A_1$ is the polynomial ring $K[u]$, and $I_k = u^{k - [\frac{k}{d+1}]} K[u]$, for all $k \geq 1$. In particular, $I_1 = uK[u]$ and $I_{i(d+1)-1} = I_{i(d+1)} = u^{id} K[u]$, for all $i \geq 1$. Hence, $I_1 I_{i(d+1)-1} \neq I_{i(d+1)}$, for all $i \geq 1$; and so the Dixmier's Problem 5 has negative answer.*

In order to prove this result we consider the Weyl algebra as the *generalized Weyl algebra* $A_1 = K[H](\sigma, H) = \oplus_{i \in \mathbb{Z}} K[H]v_i$ (see Section 2 for details). The localization of

A_1 at the Ore set $S = K[H] \setminus \{0\}$ is the skew Laurent extension $B = K(H)[X, X^{-1}; \sigma] = \bigoplus_{i \in \mathbb{Z}} K(H)X^i$ with the K -automorphism σ of the field of rational functions $K(H)$ defined by $\sigma(H) = H - 1$. The Weyl algebra A_1 is a homogeneous subalgebra of the \mathbb{Z} -graded algebra B . In Section 2, for a homogeneous element αX^i , $\alpha \in K(H)$, $i \in \mathbb{Z}$, the centralizer $C(u, B)$ (Proposition 2.1) and the algebra $N(u, B)$ (Theorem 2.3) are described. In Section 3, using these results, for an arbitrary homogeneous element u of the Weyl algebra A_1 , the centralizer $C(u, A_1)$ (Proposition 3.1) and the algebra $N(u, A_1)$ (Theorem 3.2) are found. Then, for the element $u = \alpha X$ as in the Theorem 1.1, we can describe the algebra $N(u, A_1)$ and the ideals I_n . In Section 5, we classify homogeneous elements of the Weyl algebra A_1 with respect to the Dixmier partition of elements of A_1 into the classes Δ_i . We prove (Corollary 5.3) that for an arbitrary homogeneous element u of the Weyl algebra A_1 of nilpotent type : (i) *the Dixmier's Problem 4* has positive answer, that is the associated graded algebra $\mathcal{G}(u, A_1)$ of the N -algebra $N(u, A_1)$ is an affine commutative algebra, hence Noetherian, and as a consequence the algebra $N(u, A_1)$ is affine Noetherian; (ii) the Weyl algebra A_1 is *not* a finitely generated (left and right) $N(u, A_1)$ -module.

For more information about the Weyl algebras the reader is referred to the following books [12, 22, 26].

2 Centralizer of a Homogeneous Element of the Algebra B

Let D be a ring with an automorphism σ and a central element a . The **generalized Weyl algebra** $A = D(\sigma, a)$ of degree 1, is the ring generated by D and two indeterminates X and Y subject to the relations:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y, \text{ for all } \alpha \in D, \quad YX = a \text{ and } XY = \sigma(a).$$

The algebra $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a \mathbb{Z} -graded algebra where $A_n = Dv_n$, $v_n = X^n$ ($n > 0$), $v_n = Y^{-n}$ ($n < 0$), $v_0 = 1$. It follows from the above relations that

$$v_n v_m = (n, m) v_{n+m} = v_{n+m} < n, m >$$

for some $(n, m) = \sigma^{-n-m}(< n, m >) \in D$. If $n > 0$ and $m > 0$ then

$$n \geq m : (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \quad (-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$

$$n \leq m : (n, -m) = \sigma^n(a) \cdots \sigma(a), \quad (-n, m) = \sigma^{-n+1}(a) \cdots a,$$

in other cases $(n, m) = 1$.

Let $K[H]$ be a polynomial ring in one variable H over the field K , $\sigma : H \rightarrow H - 1$ be the K -automorphism of the algebra $K[H]$ and $a = H$. The first Weyl algebra $A_1 = K \langle X, Y \mid YX - XY = 1 \rangle$ is isomorphic to the generalized Weyl algebra

$$A_1 \simeq K[H](\sigma, H), \quad X \leftrightarrow X, \quad Y \leftrightarrow Y, \quad YX \leftrightarrow H.$$

We identify both these algebras via this isomorphism, that is $A_1 = K[H](\sigma, H)$ and $H = YX$.

If $n > 0$ and $m > 0$ then

$$n \geq m : (n, -m) = (H - n) \cdots (H - n + m - 1), \quad (-n, m) = (H + n - 1) \cdots (H + n - m),$$

$$n \leq m : (n, -m) = (H - n) \cdots (H - 1), \quad (-n, m) = (H + n - 1) \cdots H,$$

in other cases $(n, m) = 1$.

The localization $B = S^{-1}A_1$ of the Weyl algebra A_1 at the Ore subset $S = K[H] \setminus \{0\}$ of A_1 is the *skew Laurent polynomial ring* $B = K(H)[X, X^{-1}; \sigma]$ with coefficients from the field $K(H) = S^{-1}K[H]$ of rational functions and $\sigma \in \text{Aut}_K K(H)$, $\sigma(H) = H - 1$. The map $A_1 \rightarrow B$, $a \rightarrow a/1$, is an algebra monomorphism. We identify the algebra A_1 with its image in the algebra B , in more detail, via the algebra monomorphism

$$A_1 \rightarrow B, \quad X \rightarrow X, \quad Y \rightarrow HX^{-1}.$$

The subalgebra $\mathcal{A} := K[H][X, X^{-1}; \sigma]$ of B contains the Weyl algebra A_1 (since the algebra generators X and $Y = HX^{-1}$ of A_1 belong to \mathcal{A}), moreover, \mathcal{A} is the localization $\mathcal{A} = A_{1,X}$ of A_1 at the powers of the element X . Clearly, $B = S^{-1}\mathcal{A}$. The algebras $B = \bigoplus_{i \in \mathbb{Z}} B_i$ and $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$ are \mathbb{Z} -graded algebras where $B_i = K(H)X^i$ and $\mathcal{A}_i = K[H]X^i$. The algebras A_1 and \mathcal{A} are \mathbb{Z} -graded subalgebras of B .

A polynomial $f(H) = \lambda_n H^n + \lambda_{n-1} H^{n-1} + \cdots + \lambda_0 \in K[H]$ of degree n is called a *monic* polynomial if the leading coefficient λ_n of $f(H)$ is 1. A rational function $h \in K(H)$ is called a *monic* rational function if $h = f/g$ for some monic polynomials f, g . A homogeneous element $u = \alpha x^n$ of B is called *monic* iff α is a monic rational function. We can extend in the obvious way the notion of degree of a polynomial to the field of rational functions setting, $\deg_H h = \deg_H f - \deg_H g$, for $h = f/g \in K[H]$. If $h_1, h_2 \in K(H)$ then $\deg_H h_1 h_2 = \deg_H h_1 + \deg_H h_2$, and $\deg_H(h_1 + h_2) \leq \max\{\deg_H h_1, \deg_H h_2\}$. We denote by $\text{sign}(n)$ and by $|n|$ the sign and the absolute value of $n \in \mathbb{Z}$, respectively.

Proposition 2.1 (CENTRALIZER OF A HOMOGENEOUS ELEMENT OF THE ALGEBRA B)

1. Let $u = \alpha X^n$ be a monic element of B_n with $n \neq 0$. The centralizer $C(u, B) = K[v, v^{-1}]$ is a Laurent polynomial ring in a uniquely defined variable $v = \beta X^{\text{sign}(n)s}$ where s is the minimal positive divisor of n for which there exists an element $\beta = \beta_s \in K(H)$, necessarily monic and uniquely defined, such that

$$\beta \sigma^s(\beta) \sigma^{2s}(\beta) \cdots \sigma^{(n/s-1)s}(\beta) = \alpha, \quad \text{if } n > 0, \quad (1)$$

$$\beta \sigma^{-s}(\beta) \sigma^{-2s}(\beta) \cdots \sigma^{-(|n|/s-1)s}(\beta) = \alpha, \quad \text{if } n < 0. \quad (2)$$

2. Let $u \in K(H) \setminus K$. Then $C(u, B) = K(H)$.

Proof. The element u is a homogeneous element of the \mathbb{Z} -graded algebra B , hence its centralizer

$$C = C(u, B) = \oplus_{i \in \mathbb{Z}} C_i, \quad C_i = C \cap B_i,$$

is a graded subalgebra of B . Consider

$$H \equiv H(u, B) := \{i \in \mathbb{Z} \mid C_i \neq 0\}.$$

The set H is a subgroup of \mathbb{Z} : $0 \in H$, since $1 \in C$; $H + H \subseteq H$, since $C_i C_j \subseteq C_{i+j}$ and B is a domain; $-H \subseteq H$, since if $0 \neq v \in C_i$, then $v^{-1} \in C_{-i}$.

1. In this case, $H = \mathbb{Z}s$ for a uniquely defined positive divisor s of n ($n \in H$, since $u \in C_n$).

Claim 1: for every $i \in H$, $C_i = K^* \alpha_i X^i$ for a uniquely defined monic $\alpha_i \in K(H)$. Obviously, $\beta X^i \in C_i$, for some $0 \neq \beta \in K(H)$, iff

$$\frac{\sigma^i(\alpha)}{\alpha} = \frac{\sigma^n(\beta)}{\beta}. \quad (3)$$

So, if $0 \neq \beta_1 X^i, \beta_2 X^i \in C_i$, then

$$\frac{\sigma^n(\beta_1)}{\beta_1} = \frac{\sigma^n(\beta_2)}{\beta_2}$$

or, equivalently, $\sigma^n(\beta_2/\beta_1) = \beta_2/\beta_1 \in K(H)^{\sigma^n} = K^*$, this finishes the proof of the claim.

It follows from the claim and from $H = \mathbb{Z}s$, that $C(u, B) = K[v, v^{-1}]$ for some v .

By the claim 1, there exists a unique monic element $0 \neq \beta \in K(H)$ such that $v = \beta X^{\text{sign}(n)s}$. If $n > 0$, then

$$C_n = K^* \alpha X^n \ni v^{n/s} = (\beta X^s)^{n/s} = \beta \sigma^s(\beta) \cdots \sigma^{(n/s-1)s}(\beta) X^n \equiv \beta_n X^n,$$

hence $\alpha = \beta_n$, since β_n is a monic polynomial. If $n < 0$, then

$$C_n = K^* \alpha X^n \ni (\beta X^{-s})^{|n|/s} = \beta \sigma^{-s}(\beta) \sigma^{-2s}(\beta) \cdots \sigma^{-(|n|/s-1)s}(\beta) X^n \equiv \beta_n X^n,$$

hence $\alpha = \beta_n$, since β_n is a monic polynomial.

Claim 2: suppose that for some positive divisor s of n and for some $0 \neq \beta \in K(H)$ one of the corresponding equalities, (1) or (2), holds. Then $\beta X^{\text{sign}(n)s} \in C$. Consider the case $n > 0$. Then

$$\frac{\sigma^s(\alpha)}{\alpha} = \sigma^s(\beta) \sigma^{2s}(\beta) \cdots \sigma^{(n/s)s}(\beta) / \beta \sigma^s(\beta) \sigma^{2s}(\beta) \cdots \sigma^{(n/s-1)s}(\beta) = \frac{\sigma^n(\beta)}{\beta},$$

hence $\beta X^s \in C$, by (3). Claim 2 proves the minimality of the s (in the Proposition).

2. The centralizer C of u is a homogeneous subalgebra of B which contains $K(H)$. A homogeneous element βX^i of B with $i \neq 0$ commutes with u iff $\beta = 0$ since $0 = [\beta X^i, u] = \beta(\sigma^i(u) - u)X^i$ and $u \notin K = K(H)^{\sigma^i}$. This proves that $C = K(H)$. ■

DEFINITION. The uniquely defined element v from Proposition 2.1.(1) is called the *canonical generator* of the algebra $C(u, B)$.

The set of polynomials

$$\varphi_0 := 1, \varphi_n := (-1)^n \frac{H(H+1) \cdots (H+n-1)}{n!} = (-1)^n \frac{H\sigma^{-1}(H) \cdots \sigma^{-n+1}(H)}{n!}, \quad n \geq 1, \quad (4)$$

is a K -basis of the polynomial algebra $K[H]$, $\deg \varphi_n = n$ and

$$\sigma(\varphi_n) - \varphi_n = \varphi_{n-1}, \text{ for all } n \geq 0, \varphi_{-1} := 0.$$

Let K be a field and let $K[t]$ be a polynomial ring in an indeterminate t . Let M be a $K[t]$ -module. For an element $p \in K[t]$ we denote by $\ker p_M$ the kernel of the K -linear map $p = p_M : M \rightarrow M, m \rightarrow pm$. The kernel $\ker p_M$ is a $K[t]$ -submodule of M . For $i \geq 0$, let $N_i = N_i(t, M) := \ker t_M^{i+1}$. Then

$$N_0 \subseteq N_1 \subseteq \cdots \subseteq N_i \subseteq \cdots, \quad tN_0 = 0, \quad \text{and} \quad tN_j \subseteq N_{j-1}, \quad \text{for } j \geq 1.$$

Clearly, $N = N(t, M) := \cup_{i \geq 0} N_i$ is a $K[t]$ -submodule of M . We set $N_{-1} = 0$. If $0 \neq u \in N$ then the unique i such that $u \in N_i \setminus N_{i-1}$ is called the *nilpotent degree* of u , denoted by $\text{ndeg } u$.

DEFINITION. A $K[t]$ -module M is called a *Jordan* $K[t]$ -module iff $M = N(t, M)$ and $tM = M$.

If M is a nonzero Jordan $K[t]$ -module then $N_i \neq N_{i+1}$ for all $i \geq 0$ since otherwise $M = N_j$ for some j , hence $M = t^{j+1}M = 0$, a contradiction. From this fact we conclude that each nonzero Jordan $K[t]$ -module is not a finitely generated module (hence, is not noetherian).

Example. A vector space $\mathcal{J} = \oplus_{i \geq 0} Ke_i$ with the $K[t]$ -module structure defined by $te_0 = 0$ and $te_i = e_{i-1}, i \geq 1$, is a Jordan module. The module \mathcal{J} is isomorphic to the $K[t]$ -module $K[t, t^{-1}]/K[t]$, and $\mathcal{J}_i := \ker t^{i+1} = \oplus_{j=0}^i Ke_j$.

Lemma 2.2 *Let M be a $K[t]$ -module. Suppose that $N(t, M)$ contains a Jordan module N' such that $N' \supseteq \ker t$. Then $N(t, M) = N'$.*

Proof. Since $N = \cup_{i \geq 0} N_i$, it suffices to show that each N_i is contained in N' . We use induction on i . The case $i = 0$ is true by the assumption. Suppose that $N_{i-1} \subseteq N'$ and let $u \in N_i$. Since $tu \in N_{i-1} \subseteq N'$ and $tN' = N'$ (N' is a Jordan module), we have $tu = tv$ for some $v \in N'$. Now, $t(u - v) = 0$, hence $u - v = w \in N_0 \subseteq N'$, and finally $u = v + w \in N'$. It means that $N_i \subseteq N'$, as required. ■

As we have seen in the Introduction, one can associate with a non-scalar element u of the Weyl algebra A_1 the inner derivation $\text{ad } u$ and the \mathbb{N} -filtered subspace $N(u) = N(u, A_1) := \cup_{i \geq 0} N(u, i, A_1)$ where $N(u, i, A_1) = \ker(\text{ad } u)^{i+1}$. The zero component of this filtration is the centralizer $C(u, A_1)$ of the element u in A_1 . The vector space $N(u, A_1)$ is in fact a \mathbb{N} -graded algebra as follows from the formula,

$$(\text{ad } u)^n(ab) = \sum_{i=0}^n \binom{n}{i} (\text{ad } u)^i(a)(\text{ad } u)^{n-i}(b), \quad (5)$$

where $a, b \in A_1$ and $n \geq 1$. The algebra $N(u, A_1)$ is a $K[\text{ad } u]$ -module, such that

$$(\text{ad } u)^i N(u, j, A_1) \subseteq N(u, j - i, A_1), \quad \text{for all } i, j \geq 0,$$

where we set $N(u, -i, A_1) = 0$ for $i \geq 1$. Each $N(u, i, A_1)$ is a finitely generated $C(u, A_1)$ -module (Proposition 10.2.(ii), [16]). The associated graded algebra of the \mathbb{N} -graded algebra $N(U, A_1)$,

$$\mathcal{G}(u, A_1) := \bigoplus_{i \geq 0} N(u, i, A_1) / N(u, i - 1, A_1),$$

is a commutative domain (Proposition 10.2.(i), [16]).

Dixmier's Problem 4, [16]: *is the algebra $\mathcal{G}(u, A_1)$ finitely generated?*

This problem is still open. In the next section a positive answer will be obtained for all homogeneous elements of the Weyl algebra. But first, we need a description of the algebra $N(u, B)$ for an arbitrary homogeneous element of the algebra B .

Theorem 2.3 (THE ALGEBRA $N(u, B)$ OF A HOMOGENEOUS ELEMENT OF THE ALGEBRA B)

1. Let $u = \alpha X^n$ where $0 \neq n \in \mathbb{Z}$ and α is a nonzero monic element of $K(H)$.
 - (i) The algebra $N(u, B)$ is generated by the algebra $C(u, B)$ and the element H . If $C(u, B) = K[v, v^{-1}]$ where $v = \beta X^t$ is chosen as in Proposition 2.1 then the algebra $N(u, B)$ is the skew Laurent extension $K[H][v, v^{-1}; \sigma^t]$. So, $N(u, B)$ is an affine Noetherian algebra.
 - (ii) For $j \geq 0$, $N(u, j, B) = \sum_{i=0}^j C(u, B) H^i$ and $H^j \in N(u, j, B) \setminus N(u, j - 1, B)$.
 - (iii) The associated graded algebra $\mathcal{G}(u, B)$ is the polynomial algebra $C(u, B)[h]$ with coefficients from $C(u, B)$ where $h := H + C(u, B) \in N(u, 1, B) \setminus C(u, B)$. Hence $\mathcal{G}(u, B)$ is an affine commutative algebra.
2. Let $u \in K(H) \setminus K$. Then $N(u, B) = C(u, B) = K(H)$.

Proof. 1. Clearly, $H \in N(u, 1, B) \setminus C(u, B)$ since

$$[u, H] = (\sigma^n(H) - H)u = -nu$$

is a nonzero element of $C(u, B)$. Denote by N' the subalgebra of B generated by the algebra $C(u, B)$ and the element H . The algebra N' is a homogeneous subalgebra of the \mathbb{Z} -graded algebra $B = \bigoplus_{i \in \mathbb{Z}} K(H) X^i$ since N' is generated by the homogeneous elements v, v^{-1} and H of B . Using this fact we see that

$$N' = K[H][v, v^{-1}; \sigma^t]$$

is a skew Laurent polynomial ring with coefficients from the polynomial ring $K[H]$. We aim to show that $N' = N(u, B)$. The inclusion $N' \subseteq N(u, B)$ is obvious since the algebra

generators v , v^{-1} and H of N' belong to $N(u, B)$. By Proposition 2.1.(1), $u = v^k$ where $k = t^{-1}n$ is a natural number. The set of elements

$$\varphi_i(n^{-1}H)v^j, \quad i \geq 0, \quad j \in \mathbb{Z}, \quad (6)$$

is a K -basis of N' where the polynomials $\varphi_i = \varphi_i(H)$ are defined in (4), and

$$\text{ad } u(\varphi_i(n^{-1}H)v^j) = \varphi_{i-1}(n^{-1}H)v^{j+k}.$$

This means that N' is a Jordan $K[\text{ad } u]$ -module which contains $\ker \text{ad } u = C(u, B)$ and $N' \subseteq N(u, B)$. By Lemma 2.2, $N' = N(u, B)$.

It follows from (6) that $N(u, j, B) = \sum_{i=0}^j C(u, B)\varphi_i(n^{-1}H) = \sum_{i=0}^j C(u, B)H^i$ and that $H^j \in N(u, j, B) \setminus N(u, j-1, B)$. So, we have proved the statements (i) and (ii). Statement (iii) is evident.

2. By the assumption, $u \in K(H) \setminus K$, thus, for each nonzero $i \in \mathbb{Z}$, the element $\sigma^i(u) - u$ is nonzero (since $K(H)^{\sigma^i} = K$, the algebra of σ^i -invariant elements in the field $K(H)$). Let $w = \alpha X^m$ be a nonzero homogeneous element of B . For $i \geq 1$,

$$(\text{ad } u)^i w = (u - \sigma^m(u))^i w \neq 0.$$

It follows easily from this fact that $N(u, B) = K(H)$. ■

3 Centralizer and $N(u, A_1)$ of a Homogeneous Element of the Weyl Algebra

In this section, for an arbitrary homogeneous element u of the Weyl algebra A_1 , algebra generators are found for the algebras $C(u, A_1)$ (Proposition 3.1) and $N(u, A_1)$ (Theorem 3.2). For certain homogeneous elements of A_1 their centralizers were described in Proposition 5.3, [16]. We shall see that the Dixmier's Problem 4 has positive answer for all homogeneous elements of the Weyl algebra (Theorem 3.2).

Consider an element $u = \alpha v_n \in A_1$ with $n \neq 0$ and a nonzero monic polynomial α of $K[H]$. If $n > 0$ then $u = \alpha X^n$, and if $n < 0$ then $u = \alpha Y^n$. The element u is a monic homogeneous element of the algebra B since α is a monic polynomial and

$$Y^n = Y^n X^n X^{-n} = (-n, n)X^{-n} = H(H+1) \cdots (H+n-1)X^{-n} = (-1)^n n! \varphi_n X^{-n}. \quad (7)$$

By Proposition 2.1.(1), $C(u, B) = K[v, v^{-1}]$ where $v = \beta X^t$ is the canonical generator of the algebra $C(u, B)$, $0 \neq \beta \in K(H)$ and the integer t has the same sign as n . Moreover,

$$v^m = u, \quad m = t^{-1}n \geq 1. \quad (8)$$

The centralizer $C(u, A_1) = A_1 \cap C(u, B) = \bigoplus_{i \in H} K v^i$ where $H = \{i \in \mathbb{Z} : v^i \in A_1\}$. By [1] or (Theorem 4.2, [16]), the algebra $C(u, A_1)$ is a finitely generated $K[u]$ -module. Since $u = v^m$ for some $m \geq 1$, using the graded argument we have $H = \{i \geq 0 : v^i \in A_1\}$.

For $i = 0, 1, \dots, m-1$, we denote by γ_i a monic polynomial of $K[H]$ of minimal possible degree, say d_i , such that $\gamma_i v^i \in A_1$. We denote by δ the inner derivation $\text{ad } u$ of the Weyl algebra A_1 . Then

$$A_1 \ni \delta^{d_i}(\gamma_i v^i) = (-n)^{d_i} d_i! v^i u^{d_i} = (-n)^{d_i} d_i! v^{i+d_i m}.$$

Thus, we can define the following non-negative integers,

$$\mu_i := \min\{j \geq 0 \mid v^j \in A_1, j \equiv i \pmod{m}\}, \text{ for each } i = 0, 1, \dots, m-1.$$

Then

$$H = \cup_{i=0}^{m-1} \{\mu_i + m\mathbb{N}\}, \quad (9)$$

a disjoint union. The next result describes the centralizer of an arbitrary homogeneous element of the Weyl algebra A_1 .

Proposition 3.1 (CENTRALIZER OF A HOMOGENEOUS ELEMENT OF THE WEYL ALGEBRA)

1. Let $u = \alpha v_n \in A_1$ where $0 \neq n \in \mathbb{Z}$ and α is a monic polynomial of $K[H]$. Then $C(u, A_1) = \oplus_{i=0}^{m-1} K[u]v^{\mu_i}$.
2. Let $u \in K[H] \setminus K$. Then $C(u, A_1) = K[H]$.

Proof. 1. By (9), $C(u, A_1) = \oplus_{j \in H} K v^j = \oplus \{K v^j \mid i = 0, \dots, m-1 \text{ and } j \in \mu_i + m\mathbb{N}\} = \oplus_{i=0}^{m-1} K[u]v^{\mu_i}$ since $v^m = u$.

2. By Proposition 2.1.(2), $C(u, B) = K(H)$, thus $C(u, A_1) = A_1 \cap C(u, B) = K[H]$. ■

Observe that X^t is equal to v_t if $t > 0$, and to $(t, -t)^{-1} v_t = (H(H+1) \cdots (H-t-1))^{-1} v_t$ if $t < 0$ (by (7)). Thus the canonical generator v can be written in the form γv_t where $\gamma = \beta$ if $t > 0$, and $\gamma = \beta(t, -t)^{-1}$ if $t < 0$. The element γ is a monic element of $K(H)$. Set $\mu := \max\{\mu_0, \dots, \mu_{m-1}\}$. Then

$$v^i = \gamma \sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma) v_{it} \in A_1, \text{ for all } i \geq \mu, \quad (10)$$

hence,

$$\gamma \sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma) \in K[H], \text{ for all } i \geq \mu. \quad (11)$$

For each $i = 1, \dots, \mu-1$, there exists a unique monic polynomial $g_i \in K[H]$ of minimal possible degree such that $g_i v^i \in A_1$. The polynomial g_i is the *denominator* of the rational function $\gamma \sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma)$ multiplied by a proper nonzero scalar. By definition, the denominator of a rational function $\alpha = pq^{-1}$ ($p, q \in K[H]$) is q provided $\gcd(p, q) = 1$. Clearly,

$$K[H]v^i \cap K[H]v_{ti} = K[H]g_i v^i, \quad i = 1, \dots, \mu-1.$$

It follows from the equality $v_{-k} v_k = (-k, k) \in K[H]$, $k \in \mathbb{Z}$, that

$$v_k^{-1} = (-k, k)^{-1} v_{-k}.$$

Now, by (10),

$$v^{-i} = v_{it}^{-1}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma))^{-1} = \{(-it, it)\sigma^{-it}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma))\}^{-1}v_{-it}.$$

By (11),

$$v_{-it} = (-it, it)\sigma^{-it}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma))v^{-i} \in N(u, A_1), \text{ for all } i \geq \mu. \quad (12)$$

For each $i \geq 1$, there exists a unique monic polynomial $f_i \in K[H]$ such that

$$K[H]v^{-i} \cap K[H]v_{-it} = (f_i)v_{-it},$$

where $(f_i) = f_i K[H]$. By (12), $f_i = 1$ for all $i \geq \mu$. For each $i = 1, \dots, \mu - 1$, the polynomial f_i is the denominator of the rational function $(-it, it)\sigma^{-it}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma))$ multiplied by a proper nonzero scalar.

Let $R = \cup_{i \in \mathbb{N}} R_i$ be an \mathbb{N} -graded algebra and $\text{gr } R = \oplus_{i \in \mathbb{N}} R_i/R_{i-1}$ be its associated graded algebra. Denote by $\pi : R \rightarrow \text{gr } R$ the principal symbol map defined $\pi(r) = r + R_{i-1}$ where $r \in R_i \setminus R_{i-1}$.

DEFINITION. A basis $E = \{e_j, j \in J\}$ of the algebra R is called a *principal* basis iff the set $\pi(E) = \{\pi(e_j), j \in J\}$ is a basis of the associated graded algebra $\text{gr } R$.

Suppose that $F = \{f_l, l \in L\}$ is a basis of the algebra $\text{gr } R$ such that each element f_l is a homogeneous element of the algebra $\text{gr } R$. For each f_l we fix its preimage e_l under the principal symbol map π , that is $\pi(e_l) = f_l$. Then the set $E = \{e_l, l \in L\}$ is a principal basis of the algebra R .

The next theorem describes the algebra $N(u, A_1)$ for an arbitrary homogeneous element of the Weyl algebra A_1 , and gives a positive answer to the Dixmier's Problem 4 for all such elements. In the next section, this result will lead us to a solution of the Dixmier's Problem 5.

Theorem 3.2 (THE ALGEBRA $N(u, A_1)$ OF A HOMOGENEOUS ELEMENT OF THE WEYL ALGEBRA)

1. Let $u = \alpha v_n \in A_1$ where $0 \neq n \in \mathbb{Z}$ and α is a monic polynomial of $K[H]$. We keep the notation above, then

$$(i) \ N(u, A_1) = \oplus_{i \geq \mu} K[H]v_{-it} \oplus (\oplus_{i=1}^{\mu-1} K[H]f_i v_{-it}) \oplus K[H] \oplus (\oplus_{i=1}^{\mu-1} K[H]g_i v^i) \oplus (\oplus_{i \geq \mu} K[H]v^i).$$

(ii) The set $S = \{H^k v_{-it}, H^k f_j v_{-jt}, H^k, H^k g_j v^j, H^k v^i \mid k \geq 0, i \geq \mu, j = 1, \dots, \mu - 1\}$ is a principal basis of the algebra $N(u, A_1)$, with $\text{ndeg } v_{-it} = i(|t| + \deg_H \gamma)$ for $i \geq \mu$, $\text{ndeg } f_j v_{-jt} = \deg f_j + j(|t| + \deg_H \gamma)$ and $\text{ndeg } g_j v^j = \deg_H g_j$ for $j = 1, \dots, \mu - 1$.

(iii) The algebra $\mathcal{G}(u, A_1)$ is an affine (commutative) algebra, hence Noetherian.

(iv) The algebra $N(u, A_1)$ is an affine Noetherian algebra.

2. Let $u \in K[H] \setminus K$. Then $N(u, A_1) = K[H] = C(u, A_1)$.

Proof. 1.(i). By Theorem 2.3.(1), the algebra $N(u, B) = \oplus_{i \in \mathbb{Z}} K[H]v^i$ and the Weyl algebra $A_1 = \oplus_{j \in \mathbb{Z}} K[H]v_j$ are homogeneous subalgebras of the algebra $B = \oplus_{j \in \mathbb{Z}} K[H]X^j$. So, the intersection

$$N(u, A_1) = A_1 \cap N(u, B) = \oplus_{i \in \mathbb{Z}} (K[H]v^i \cap K[H]v_{it}),$$

is a homogeneous subalgebra of the Weyl algebra A_1 . If we recall the definition of the polynomials f_i and g_i then the result follows immediately from the fact above and (10), (12).

(ii) Since $N(u, A_1)$ is a homogeneous subalgebra of A_1 , it is easy to see that the set S is a principal basis of $N(u, A_1)$. Since $\text{ndeg } H = 1 = \deg_H H$ and

$$v_{-it} = (-it, it)\sigma^{-it}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma))v^{-i}, \text{ for all } i \geq \mu,$$

we have $\text{ndeg } v_{-it} = \deg_H (-it, it)\sigma^{-it}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma)) = i(|t| + \deg_H \gamma)$. For each $j = 1, \dots, \mu - 1$,

$$f_j v_{-jt} = f_j(-jt, jt)\sigma^{-jt}(\gamma\sigma^t(\gamma) \cdots \sigma^{(j-1)t}(\gamma))v^{-j},$$

hence, $\text{ndeg } f_j v_{-jt} = \deg_H f_j(-jt, jt)\sigma^{-jt}(\gamma\sigma^t(\gamma) \cdots \sigma^{(j-1)t}(\gamma)) = \deg f_j + j(|t| + \deg_H \gamma)$. The rest is obvious.

(iii) Denote by R the subalgebra of $\mathcal{G} = \mathcal{G}(u, A_1)$ generated by the principal symbols of the elements $v_{-\mu t}$, H and v^μ . The set S is a principal basis of the algebra $N(u, A_1)$, thus the set $\pi(S) = \{\pi(s), s \in S\}$ is a basis of the algebra \mathcal{G} . The algebra \mathcal{G} is affine since it is a finitely generated R -module with generators which are the principal symbols of the elements

$$v_{-it}, f_j v_{-jt}, 1, g_j v^j, \text{ and } v^{it}, \text{ where } i = \mu + 1, \dots, 2\mu - 1; j = 1, \dots, \mu - 1.$$

(iv) The algebra $N(u, A_1)$ is a Noetherian affine algebra since the algebra \mathcal{G} is so.

2. By Theorem 2.3.(2), $N(u, A_1) = A_1 \cap N(u, B) = A_1 \cap K(H) = K[H] = C(u, A_1)$. ■

Let $u = \alpha v_n \in A_1$ be as in Theorem 3.2.(1). The algebra $N(u, A_1)$ is a \mathbb{Z} -graded algebra with zero graded component $K[H]$, hence the (left and right) Krull dimension (in the sense of Rentschler and Gabriel, [27]),

$$\text{K.dim } N(u, A_1) \geq \text{K.dim } K[H] = 1. \quad (13)$$

The homogeneous subalgebra A of $N(u, A_1)$ generated by the homogeneous elements $y := v_{-\mu t}$, H and $x := v^\mu$ is the generalized Weyl algebra

$$A = K[H](\sigma^{\mu t}, a := (-\mu t, \mu t)\sigma^{-\mu t}(\gamma\sigma^t(\gamma) \cdots \sigma^{(\mu-1)t}(\gamma))),$$

since, by (12),

$$xH = \sigma^{\mu t}(H)x, yH = \sigma^{-\mu t}(H)y, yx = a, \text{ and } xy = \sigma^{\mu t}(a).$$

In the terminology of [20], [15], the algebra A is called a *noncommutative deformation of type-A Kleinian singularity*. The algebra A is a (left and right) Noetherian algebra, [4], [20].

Corollary 3.3 *Let $u = \alpha v_n \in A_1$ be as in Theorem 3.2.(1).*

1. *The algebra $N(u, A_1)$ is a finitely generated A -module.*
2. *The (left and right) Krull dimension of the algebra $N(u, A_1)$ is 1.*
3. *If $\deg_H \alpha > 0$ then the Weyl algebra A_1 is not a finitely generated (left and right) $N(u, A_1)$ -module.*

Proof. 1 and 2. By Theorem 3.2.(1), the algebra $N(u, A_1)$ is a finitely generated (left and right) A -module, hence $\text{K.dim } N(u, A_1) \leq \text{K.dim } A$. Since $a \neq 0$ and $\text{char } K = 0$, the Krull dimension of the generalized Weyl algebra A is 1 (see [4] or [20]), hence $\text{K.dim } N(u, A_1) = 1$, by (13).

Since $N(u, A_1)$ is a finitely generated A -module, the Weyl algebra A_1 is not a finitely generated $N(u, A_1)$ -module iff it is not a finitely generated A -module. So, it suffices to prove that A_1 is not a finitely generated A -module. By (8) and (10), $\alpha v_n = u = \gamma \sigma^t(\gamma) \cdots \sigma^{(m-1)t}(\gamma) v_n$, hence $0 < \deg_H \alpha = \deg_H \gamma \sigma^t(\gamma) \cdots \sigma^{(m-1)t}(\gamma) = m \deg_H \gamma$, thus $\deg_H \gamma > 0$ and $\deg_H a > 0$. It suffices to show that the factor module A_1/A is not a finitely generated A -module. Observe that A is a homogeneous subalgebra of A_1 , and that

$$M := (\oplus_{i \in \mathbb{Z}} K[H]v_{i\mu t})/A = \oplus_{i \geq 1} (K[H]v_{i\mu t}/K[H]v^{i\mu})$$

is an \mathbb{N} -graded A -submodule of A_1/A . The i 'th component of M , $M_i := K[H]v_{i\mu t}/K[H]v^{i\mu}$, as a $K[H]$ -module, is canonically isomorphic to

$$K[H]v_{i\mu t}/K[H]\gamma_{i\mu}v_{i\mu t} \simeq K[H]/K[H]\gamma_{i\mu},$$

where $\gamma_{i\mu} := \gamma \sigma^t(\gamma) \cdots \sigma^{(i\mu-1)t}(\gamma)$. Since $\gamma_{i\mu}M_i = 0$, for all $i \geq 1$, the $K[H]$ -module M_i is a $K[H]$ -torsion module. Each finitely generated $K[H]$ -torsion module over the generalized Weyl algebra A has finite length and Gelfand-Kirillov dimension ≤ 1 ([4, 6]).

Suppose that M is a finitely generated A -module, then $\text{GK}(M) = 1$ since $\dim_K M = \infty$. The algebra A is a *somewhat commutative* algebra, [20], hence there exists a natural number c such that

$$\sum_{i=1}^n \dim M_i \leq cn \text{ for all } n \gg 0,$$

which contradicts $\sum_{i=1}^n \dim M_i = \sum_{i=1}^n \deg_H \gamma_{i\mu} = \sum_{i=1}^n i\mu \deg_H \gamma = \mu \deg_H(\gamma) \frac{n(n+1)}{2}$. Thus M is not a finitely generated A -module. ■

4 Solution to the Dixmier's Problem 5

In this section we apply the results from the previous sections to show that the Dixmier's Problem 5 has *negative solution*.

Lemma 4.1 *Let α be a monic polynomial of $K[H]$ of degree $d \geq 1$ and $u = \alpha X \in A_1$. Then*

1. $C(u, A_1) = K[u]$.
2. $N(u, A_1) = \oplus_{i \geq 1} K[H]Y^i \oplus \oplus_{i \geq 0} K[H]u^i$ and the set $\{\varphi_i Y^{j+1}, \varphi_i u^j \mid i, j \geq 0\}$ is a basis of the algebra $N(u, A_1)$.
3. $Y \in N(u, d+1, A_1) \setminus N(u, d, A_1)$ and, for $k \geq 1$, $N(u, k, A_1) = \oplus_{i, j \geq 0} \{K\varphi_i Y^j \mid i + (d+1)j \leq k\} \oplus \oplus_{i=0}^k K[u]\varphi_i$.

Proof. 1. By Proposition 2.1.(1), the algebra $C(u, B) = K[u, u^{-1}]$, hence, by Proposition 3.1.(1), we have $C(u, A_1) = K[u]$.

2. The element u is the canonical generator of the algebra $C(u, B) = K[u, u^{-1}]$. Since $\alpha \in K[H]$, by Theorem 3.2.(1), $N(u, A_1) = \oplus_{i \geq 1} K[H]Y^i \oplus \oplus_{i \geq 0} K[H]u^i$. The rest is evident.

3. By Theorem 2.3, we have $H \in N(u, 1, A_1) \setminus N(u, 0, A_1)$, hence

$$Y = HX^{-1} = H\sigma^{-1}(\alpha)(\alpha X)^{-1} = H\sigma^{-1}(\alpha)u^{-1} \in N(u, d+1, A_1) \setminus N(u, d, A_1),$$

and

$$\varphi_i(H)Y^j \in N(u, i + (d+1)j, A_1) \setminus N(u, i + (d+1)j - 1, A_1), \text{ for all } i, j \geq 0.$$

Now, the result follows from Statement 2. ■

Let $u = \alpha X$ be as in Lemma 4.1. We denote by δ the inner derivation $\text{ad } u$ of the Weyl algebra A_1 .

For each $i \geq 1$, $\delta(\varphi_i) = (\sigma(\varphi_i) - \varphi_i)u = \varphi_{i-1}u$, hence

$$\delta^i(\varphi_i) = u^i. \tag{14}$$

Clearly,

$$\begin{aligned} Y^i &= (HX^{-1})^i = H(H+1) \cdots (H+i-1)X^{-i} \\ &= H(H+1) \cdots (H+i-1)\sigma^{-1}(\alpha)\sigma^{-2}(\alpha) \cdots \sigma^{-i}(\alpha)u^{-i} \\ &= (H^{i(d+1)} + \cdots)u^{-i} = (-1)^{i(d+1)}[i(d+1)]! \varphi_{i(d+1)}u^{-i} + \cdots, \end{aligned}$$

where by three dots we denote, as usually, elements of smaller nilpotent degree. So,

$$\delta^{(d+1)i}(Y^i) = (-1)^{i(d+1)}[(d+1)i]! u^{id}. \tag{15}$$

Using (5), we have

$$\delta^{i+(d+1)j}(\varphi_i Y^j) = \binom{i+(d+1)j}{i} \delta^i(\varphi_i) \delta^{(d+1)j}(Y^j) = (-1)^{j(d+1)} \binom{i+(d+1)j}{i} [(d+1)j]! u^{i+dj}, \tag{16}$$

for all $i, j \geq 0$.

Corollary 4.2 (SOLUTION TO THE DIXMIER'S PROBLEM 5) *Let $u = \alpha X$ be as in Lemma 4.1. Then $I_k = u^{k - [\frac{k}{d+1}]} K[u]$, for all $k \geq 1$. In particular, $I_1 = uK[u]$ and $I_{i(d+1)-1} = I_{i(d+1)} = u^{id} K[u]$, for all $i \geq 1$. Hence, $I_1 I_{i(d+1)-1} \neq I_{i(d+1)}$, for all $i \geq 1$, and the Dixmier's Problem 5 has negative solution.*

Proof. By Lemma 4.1.(3) and (16), $I_k = u^{k - [\frac{k}{d+1}]} K[u]$, for all $k \geq 1$. The rest is obvious. ■

It turns out that, for the element u as above, the algebra $N(u, A_1)$ is a generalized Weyl algebra of a special sort. So, applying the results of the papers [4]–[6], [20], where these algebras were studied, we can say a lot about them. We collect some of the results in the following corollary.

Corollary 4.3 *Let $u \in A_1$ be as in Lemma 4.1. Then*

1. *The algebra $N(u, A_1)$ is a generalized Weyl algebra $K[H](\sigma, H\sigma^{-1}(\alpha))$, a so-called noncommutative deformation of type A-Kleinian singularity in the terminology of [20], [15].*
2. *The algebra $N(u, A_1)$ is simple iff, for any two distinct monic irreducible factors p and q from $K[H]$ of the polynomial $H\sigma^{-1}(\alpha)$, there is no an integer i such that $\sigma^i(p) = q$.*
3. *The algebra $N(u, A_1)$ has only finitely many (two-sided) ideals, they are classified in [5]. Each nonzero ideal has finite codimension in $N(u, A_1)$.*
4. *The Krull dimension of the algebra $N(u, A_1)$ is 1.*
5. *Let $H\sigma^{-1}(\alpha) = p_1^{n_1} \cdots p_s^{n_s}$ be a product of distinct monic irreducible polynomials. The global dimension*

$$\text{gl.dim } N(u, A_1) = \begin{cases} \infty & , \text{ if there exists } n_i \geq 2; \\ 2 & , \text{ if } n_1 = \cdots = n_s = 1 \text{ and } \sigma^i(p_j) = p_k \\ & \text{for some } j \neq k \text{ and some integer } i; \\ 1 & , \text{ otherwise.} \end{cases}$$

Proof. 1. By Lemma 4.1.(2), the algebra $N(u, A_1)$ is generated by the elements Y , H and $X' = \alpha X$. Since

$$X'H = \sigma(H)X', \quad YH = \sigma^{-1}(H)Y, \quad YX' = H\sigma^{-1}(\alpha) \text{ and } X'Y = \sigma(H\sigma^{-1}(\alpha)),$$

the algebra $N(u, A_1)$ is isomorphic to the generalized Weyl algebra $K[H](\sigma, H\sigma^{-1}(\alpha))$ in a view of the decomposition from Lemma 4.1.(2).

2 and 3. These results were proved in [4, 5, 6].

4 and 5. These results were proved in [4, 6, 20]. ■

Corollary 4.4 *Let $u \in A_1$ be as in Lemma 4.1. Then the left $N(u, A_1)$ -module $M = A_1/N(u, A_1)$ is a $K[H]$ -torsion, not finitely generated left $N(u, A_1)$ -module of Gelfand-Kirillov dimension 1. Each finitely generated $N(u, A_1)$ -submodule of M has finite length. The set of isomorphism classes of simple subfactors of all finitely generated $N(u, A_1)$ -submodules of M is a finite set.*

Proof. The algebra $N(u, A_1)$ is a homogeneous subalgebra of the Weyl algebra A_1 , thus the $N(u, A_1)$ -module $M = \bigoplus_{i \geq 1} M_i$ is an \mathbb{N} -graded $N(u, A_1)$ -module where the i 'th component M_i , as a $K[H]$ -module, is canonically isomorphic to

$$K[H]/u^i K[H] = K[H]/\alpha_i X^i K[H] \simeq K[H]/\alpha_i K[H],$$

where $\alpha_i := \alpha \sigma(\alpha) \cdots \sigma^{i-1}(\alpha)$. Since $\alpha_i M_i = 0$, for all $i \geq 1$, the module M is a $K[H]$ -torsion module. Each finitely generated $K[H]$ -torsion module over a generalized Weyl algebra of the type $K[H](\sigma, a \neq 0)$, for example $N(u, A_1)$, has finite length and Gelfand-Kirillov dimension ≤ 1 , thus $\text{GK}(M) \leq 1$. Observe that the $N(u, A_1)$ -submodule L of M generated by the element $\bar{X} = X + N(u, A_1)$ is not finite dimensional since

$$u^i \bar{X} = \alpha_i X^i \bar{X} = X \sigma^{-1}(\alpha_i) X^i + N(u, A_1) = X(\sigma^{-1}(\alpha_i) - \alpha_i) X^i + N(u, A_1) = (\alpha_i - \sigma(\alpha_i)) \bar{X}^i \neq 0$$

since $0 < \deg(\alpha_i - \sigma(\alpha_i)) < \deg \alpha_i$. So, $1 \leq \text{GK}(L) \leq \text{GK}(M) \leq 1$, hence $\text{GK}(M) = 1$.

A finitely generated $N(u, A_1)$ -submodule, say V , of M is a submodule of the module U_s generated by $\bigoplus_{i=1}^s M_i$ for some s . The $N(u, A_1)$ -module U_s is an epimorphic image of the $N(u, A_1)$ -module $\bigoplus_{i=1}^s N(u, A_1)/N(u, A_1)\alpha_i$. Each $N(u, A_1)$ -module $N(u, A_1)/N(u, A_1)\alpha_i$ has finite length, and the set of all isomorphic classes of all simple subfactors of all modules $N/N\alpha_i$ is a finite set since $\alpha_i := \alpha \sigma(\alpha) \cdots \sigma^{i-1}(\alpha)$ (see [4, 6] for details). Now the result follows. ■

5 Classification of Homogeneous Elements of the Weyl Algebra and the Dixmier's Problem 4

Let u be a non-scalar element of the Weyl algebra A_1 . The corresponding inner derivation $\text{ad } u$ of A_1 is denoted by δ . Denote by $\text{Ev}(\text{ad } u, A_1) = \text{Ev}(u, A_1)$ the set of all eigenvalues of the linear map δ acting in the vector space A_1 . For an eigenvalue λ of δ we denote by $D(u, \lambda, A_1)$ the set of all eigenvectors of δ with the eigenvalue λ . The map δ is a derivation of the Weyl algebra A_1 , so the set $\text{Ev}(u, A_1)$ is an additive submonoid of the field K , and the vector space

$$D(u)^{ev} = D(u, A_1)^{ev} := \bigoplus_{\lambda \in \text{Ev}(u, A_1)} D(u, \lambda, A_1)$$

is a $\text{Ev}(u, A_1)$ -graded algebra, that is,

$$D(u, \lambda, A_1) D(u, \mu, A_1) \subseteq D(u, \lambda + \mu, A_1), \quad \text{for all } \lambda, \mu \in \text{Ev}(u, A_1).$$

Let a field \bar{K} be an algebraic closure of the field K . The tensor product of algebras $\bar{A}_1 = K \otimes A_1$ over the field K is the Weyl algebra over the field \bar{K} which contains the Weyl K -algebra A_1 . Then

$$D(u) = D(u, A_1) := A_1 \cap D(u, \bar{A}_1)^{ev}$$

is a K -subalgebra of A_1 which contains the algebra $D(u)^{ev}$ but does not necessarily coincide with this algebra. The next result, obtained by Dixmier, [16], classifies non-scalar elements of the Weyl algebra A_1 with respect to the properties of the corresponding inner derivations of elements.

Theorem 5.1 (THE DIXMIER'S CLASSIFICATION OF NON-SCALAR ELEMENTS OF THE WEYL ALGEBRA) *The set of non-scalar elements of the Weyl algebra A_1 is a disjoint union of the following subsets:*

- (i) $\Delta_1 = \{x \in A_1 \setminus K : N(x) = A_1, D(x) = C(x)\}.$
- (ii) $\Delta_2 = \{x \in A_1 \setminus K : N(x) \neq A_1, N(x) \neq C(x), D(x) = C(x)\}.$
- (iii) $\Delta_3 = \{x \in A_1 \setminus K : D(x) = A_1, N(x) = C(x)\}.$
- (iv) $\Delta_4 = \{x \in A_1 \setminus K : D(x) \neq A_1, D(x) \neq C(x), N(x) = C(x)\}.$
- (v) $\Delta_5 = \{x \in A_1 \setminus K : D(x) = N(x) = C(x)\}.$ ■

Each subset Δ_i of A_1 is a non-empty set.

DEFINITION. Elements of $\Delta_3 \cup \Delta_4$ (resp. of Δ_3) are called elements of *semi-simple type* (resp. of *strongly semi-simple type*). Dixmier classified elements of strongly semi-simple type, Theorem 9.2, [16]: $x \in \Delta_3$ iff there exists an automorphism $\tau \in \text{Aut}_K A_1$ such that $\tau(x) = \lambda Y^2 + \mu X^2 + \nu$ for some scalars λ, μ and ν such that $\lambda \neq 0$ and $\mu \neq 0$. It can be easily seen that, if the polynomial $\lambda t^2 + \mu$ has a root in the field K then there exists an automorphism $\tau_1 \in \text{Aut}_K A_1$ such that $\tau_1(x) = \alpha H + \beta$ for some scalars $0 \neq \alpha$ and β (see Corollary 9.3, [16]).

Theorem 5.2 (CLASSIFICATION OF HOMOGENEOUS ELEMENTS OF THE WEYL ALGEBRA) *Let $u = \alpha v_i$, $\alpha \in K[H]$, $i \in \mathbb{Z}$, be a homogeneous non-scalar element of A_1 . Then $u \in \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_5$. In more detail,*

1. $u \in \Delta_1 \Leftrightarrow \alpha \in K^*$ and $i \neq 0$.
2. $u \in \Delta_2 \Leftrightarrow \alpha \notin K$ and $i \neq 0$.
3. $u \in \Delta_3 \Leftrightarrow \deg_H \alpha = 1$ and $i = 0$.
4. $u \in \Delta_5 \Leftrightarrow \deg_H \alpha > 1$ and $i = 0$.

Proof. If $\alpha \in K^*$ and $i \neq 0$ then by [16], Proposition 10.3, $N(u, A_1) = N(v_{\pm 1}^{[i]}, A_1) = N(v_{\pm 1}, A_1) = A_1$, hence $u \in \Delta_1$. If $\alpha \notin K$ and $i \neq 0$ then $u \in \Delta_2$, by Theorem 3.2.(1). If $\deg_H \alpha = 1$ and $i = 0$, i.e. $u = \lambda H + \mu$ for some scalars $\lambda \neq 0$ and μ , then $u \in \Delta_3$. If $\deg_H \alpha > 1$ and $i = 0$ then, by Theorem 3.2.(2), $N(u, A_1) = C(u, A_1) = K[H]$. The element $u = \alpha$ is a homogeneous element of the algebra A_1 , thus the algebra

$D(u, A_1)$ is a homogeneous subalgebra of A_1 . Suppose that $D(u, A_1) \neq C(u, A_1)$ then there exists a homogeneous element βv_m ($\beta \in K[H]$) of A_1 and a nonzero scalar λ such that $\beta v_m \in D(u, \lambda, A_1)$, hence $\lambda \beta v_m = [\alpha, \beta v_m] = (\alpha - \sigma^m(\alpha))\beta v_m$. So, $\lambda = \alpha - \sigma^m(\alpha)$, hence $\deg_H \alpha = 1$, a contradiction. This means that $D(u, A_1) = C(u, A_1)$ and $u \in \Delta_5$. This finishes the proof of the theorem. ■

Corollary 5.3 *Let u be a homogeneous element of weakly nilpotent type of the Weyl algebra A_1 , i.e. $u \in \Delta_2$. Then*

1. *The associated graded algebra $\mathcal{G}(u, A_1)$ is an affine commutative algebra (thus, the Dixmier's Problem 4 has a positive answer for homogeneous elements of nilpotent type) hence the algebra $N(u, A_1)$ is an affine Noetherian algebra.*
2. *The algebra A_1 is not a finitely generated (left and right) $N(u, A_1)$ -module.*

Proof. 1. By Theorem 5.2.(2), each homogeneous element of nilpotent type of the Weyl algebra A_1 has the form as in Theorem 2.3.(1). Now observe that the result is already was proved in Theorem 5.2.(2).(iii) and (iv).

2. This follows from Corollary 3.3.(3) and Theorem 5.2.(2). ■

Let a and p be nonzero elements of the Weyl algebra A_1 satisfying $[a, p] = \lambda p$ for some $0 \neq \lambda \in K$. Then the element p is of nilpotent type, and $[a, cp] = \lambda cp$ for all nonzero elements $c \in C(a, A_1)$. The next result shows how the type of the element p changes when p is multiplied by an element from $C(a, A_1)$.

Corollary 5.4 *Let $a \in \Delta_3(A_1)$ and $[a, p] = \lambda p$ for some $0 \neq \lambda \in K$ and $p \in A_1$. Then $C(a, A_1) = K[a]$.*

1. *Suppose that $p \in \Delta_1(A_1)$ and $0 \neq \alpha(t) \in K[t]$. Then*
 - (i) $\alpha(a)p \in \Delta_1(A_1)$ *if and only if* $\alpha \in K^*$.
 - (ii) $\alpha(a)p \in \Delta_2(A_1)$ *if and only if* $\alpha \notin K$.
2. *Suppose that $p \in \Delta_2(A_1)$. Then $\alpha(a)p \in \Delta_2(A_1)$ for all nonzero polynomials $\alpha(t) \in K[t]$.*

Proof. The fact that $C(a, A_1) = K[a]$ easily follows from Theorem 9.2 and Corollary 9.4, [16].

We may assume that K is an algebraically closed field. Then, by Theorem 9.2 and Corollary 9.4, [16], there exists an automorphism $\nu \in \text{Aut}_K(A_1)$ such that $\nu(a) = \mu H + \gamma$ for some scalars $\mu \neq 0$ and γ . So, multiplying the element a by μ^{-1} and adding an appropriate scalar to H , without loss of generality we may assume that $a = H$. Now, p is a homogeneous nonscalar element of the algebra A_1 , and the result follows from Theorem 5.2. ■

ACKNOWLEDGMENT

The author would like to thank J. Dixmier and T. Lenagan for comments.

References

- [1] S. A. Amitsur, Commutative linear differential operators, *Pacific J. Math.* **8** (1958), 1–10.
- [2] J. Alev and F. Dumas, Sur les invariants des algbres de Weyl et de leurs corps de fractions. In *Rings, Hopf algebras, and Brauer groups* (Antwerp/Brussels, 1996), 1–10, Lecture Notes in Pure and Appl. Math., 197, Dekker, New York, 1998.
- [3] H. Bass, E. H. Connel and D. Wright, The Jacobian Conjecture: reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc. (New Series)*, **7** (1982), 287–330.
- [4] V. V. Bavula, Generalized Weyl algebras and their representations, *Algebra i Analiz* **4** (1992), no. 1, 75–97; English trans. in *St. Petersburg Math. J.* **4** (1993), no. 1, 71–92.
- [5] V. V. Bavula, Description of two-sided ideals in a class of non-commutative rings. 1, *Ukrainian Math. J.* **45** (1993), no. 2, 223–234 .
- [6] V. V. Bavula, Generalized Weyl algebras, kernel and tensor-simple algebras, their simple modules, *Proccedings of the 6th Int. Conf. on Represent. of Algebras* (V.Dlab and H.Lenzing Eds), CMS Conf. Proc., v. 14, 1993, 83–107.
- [7] V. V. Bavula, Dixmier’s Problem 6 for the Weyl Algebra (the Generic Type Problem). arXiv:math.RA/0402244.
- [8] V. V. Bavula and D. Jordan, Isomorphism problem and groups of automorphisms for generalized Weyl algebras, *Tr. AMS* **353** (2001), no. 2, 769–794.
- [9] V. V. Bavula and F. van Oystaeyen, Simple holonomic modules over the second Weyl algebra A_2 , *Adv. Math.* **150** (2000), no. 1, 80–116.
- [10] I. N. Bernstein, The analitic continuation of generalized functions with respect to a parameter, *Funct. Anal. and Appl.* **6** (1972), no. 4, 26–40.
- [11] I. Bernstein and V. Lunts, On non-holonomic irreducible \mathcal{D} -modules, *Inv. Math.* **94** (1988), 223–243.
- [12] J.-E. Björk, *Rings of differential operators*, North Holland, Amsterdam, 1979.
- [13] R. E. Block, The irreducible representations of the Lie algebra $sl(2)$ and of the Weyl algebra, *Adv. Math.* **39** (1981), 69–110.
- [14] S. C. Coutinho, d -simple rings and simple \mathcal{D} -modules. *Math. Proc. Cambridge Philos. Soc.* **125** (1999), no. 3, 405–415.
- [15] W. Crawley-Boevey and M. Holland, Noncommutative deformations of Kleinian singularities, *Duke Math. J.* **92** (1998), no. 3, 605–635.

- [16] J. Dixmier, Sur les algèbres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209–242.
- [17] J. Dixmier, Sur les algèbres de Weyl, II, *Bull. Sci. Math.* **94** (1972), no. 4, 26–40.
- [18] J. Dixmier, Représentation irréductibles des algèbres de Lie résolubles, *J. Math. pures et appl.* **45** (1966), 1–66.
- [19] I. M. Gelfand and A. A. Kirillov, Sur les corps liés aux algèbres enveloppantes des algèbres de Lie, Paris, Presses universitaires de France, *I.H.E.S.* **31** (1966), 5–19.
- [20] T. J. Hodges, Noncommutative deformations of type-*A* Kleinian singularities. *J. Algebra* **161** (1993), no. 2, 271–290.
- [21] A. Joseph, The Weyl algebra—semisimple and nilpotent elements, *Amer. J. Math.* **97** (1975), no. 3, 597–615.
- [22] G. Krause and T. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*. Revised edition. Graduate Studies in Mathematics, 22. American Mathematical Society, Providence, RI, 2000.
- [23] V. Lunts, Algebraic varieties preserved by generic flows, *Duke Math. J.* **58** (1989), no. 3, 531–554.
- [24] L. Makar-Limanov, The skew field of fractions of the Weyl algebra contains a free noncommutative subalgebra, *Comm. Algebra* **11** (1983), 2003–2006.
- [25] J. C. McConnell and J. C. Robson, Homomorphisms and extensions of modules over certain differential polynomial rings, *J. Algebra* **26** (1973), 319–342.
- [26] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*. With the co-operation of L. W. Small. J. Wiley, 1987.
- [27] R. Rentschler and P. Gabriel, Sur la dimension des anneaux est ensembles ordonnés, *C. R. Acad. Sci. Paris, Sér. A* **265** (1967), 712–715.
- [28] G. S. Rinehart, Note on the global dimension of a certain ring, *Proc. Amer. Math. Soc.* **13** (1962), 341–346.
- [29] J. E. Roos, Détermination de la dimension homologique des algèbres de Weyl, *C. R. Acad. Sci. Paris, Sér. A* **274** (1972), 23–26.
- [30] J. T. Stafford, Non-holonomic modules over Weyl algebras and enveloping algebras, *Invent. Math.* **79** (1985), 619–638.

Department of Pure Mathematics
 University of Sheffield
 Hicks Building
 Sheffield S3 7RH
 email: v.bavula@sheffield.ac.uk